J. DIFFERENTIAL GEOMETRY
33 (1991) 731-742

A SIMPLE CONSTRUCTION OF ATIYAH-SINGER CLASSES AND PIECEWISE LINEAR TRANSFORMATION GROUPS

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The main purpose of this paper is to provide a simple transparent construction of the Atiyah-Singer classes associated to semifree PL actions of finite groups with manifold fixed sets, such as the locally linear actions. These classes are so named because they enter into nonsmooth versions of the G-signature formula. We shall also see how they enter into existence and classification problems for group actions in a way that the smooth characteristic classes of [4] do not. Other papers which, in some cases implicitly, deal with these classes or some analogue of them (or their use in constructing actions) are, [6], [9], [10], [16], [18], [21], [26], [27], [30], [43]; they define them by a wide variety of techniques from homological surgery to sheaf theory to analytical methods more in the spirit of Atiyah and Singer. We hope that the present self-contained treatment in the simple contexts of semifree PL and of PL locally linear actions will help readers to understand the other treatments as well.

Actually there are some subtleties due to local linearity not present for more general PL actions. These are basically due to the fact that unrestricted coning is not permitted in this category. For instance, for $G = Z_{163}$ [40] (or $G = Z_n$ for n divisible by several primes [42]) the range of the G-signature map is different for PL locally linear actions than it is for general PL actions. Closer to the point of this paper, there is an additional obstruction for a submanifold of the sphere to be a semifree PL locally linear fixed set beyond those necessary for it to be a PL fixed set [41]. For this, the refinements obtained in this paper (not available in the cited papers) are crucial.

Thus approach is based on the analysis and comparison of classifying spaces $BSRN_k^G$ (respectively, $B\widetilde{SPL}_k^G$) for oriented equivariant PL

Received July 10, 1988 and, in revised form, January 11, 1990. The authors were partially supported by National Science Foundation grants. The first author is a John Simon Guggenheim Foundation Fellow and the second author is a Presidential Young Investigator and Alfred P. Sloan Foundation fellow.

(respectively, oriented equivariant PL locally linear) neighborhoods of codimension k of manifolds (cf. [19]). In §1 Quinn's geometric formulation of surgery [24], Sullivan's method of analyzing F/PL [34], and calculations of Wall [37] are used in proving:

Theorem 1. *$\widetilde{BSPL}_k^G \to \widetilde{BSPL}_k \times \widetilde{\mathbf{L}}_k^s(G) \otimes Z[1/|G|]$, $k > 2$, and $BSRN_k^G \to \widetilde{BSPL}_k \times \widetilde{\mathbf{L}}_k^h(G) \otimes Z[1/|G|]$, $k > 2$, are $Z[1/|G|]$ equivalences.*

In each case the first factor forgets the group action and the second is the PL Atiyah-Singer class. Away from 2, the localization theorem [3] for KO^G can be used to view these classes as, after suitable localizations, lying in $KO^G(M)$ for any PL manifold M with semifree PL G action with manifold fixed set. (This may suggest the connection to more analytic treatments.) The main application of Theorem 1 given in §2 is:

Theorem 2 (*Hard extension across homology collars*). *Let $(W^n; \partial_+, \partial_-)$ be a manifold triad with $\pi_1 W = \pi_1(\partial_+) = 0$, $n \geq 6$, and*

$$H_*(W, \partial_-; Z_{(|G|)}) = 0.$$

Suppose G acts orientation-preserving, semifreely and PL (respectively, locally linearly) on ∂_- with fixed set L of codimension larger than two. Then there is a PL *(locally linear) extension to such an action of G on W with fixed set a given properly embedded* PL *submanifold K (e.g., $K \cap \partial_- = L$) if and only if*

1. *$H_*(K, L; Z_{(|G|)}) = 0$;*
2. *$\sum(-1)^i \sigma[H_i(W, \partial_-)/H_i(K, L)] = 0 \in \widetilde{K}_0(ZG)$, where σ is the Swan homomorphism; and*
3. *the Atiyah-Singer classes of L extend to classes on K.*

Moreover if an extension exists, then extensions are classified by their Atiyah-Singer classes and an element of Wh(G).

Condition 1 is a result of Smith theory. The idea that one can often deduce converses to Smith theory was forcefully demonstrated in the outstanding paper of Jones [17] that has exerted enormous influence in transformation groups. In condition 2, σ denotes the Swan homomorphism from $(\mathbb{Z}/|G|)^* \to K_0(\mathbb{Z}[G])$ obtained as the connecting homomorphism of algebraic K-theory from the standard quadrad for analyzing $\mathbb{Z}[G]$, and the expression is essentially a calculation of a finiteness obstruction. Condition 3 is a computable bundle extension obstruction.

Theorem 2 immediately leads to a classification of orientation preserving PL locally linear semifree actions on the disk with fixed set of codimension $\neq 2$. (See the discussion in [18] of nonlocally linear actions.

Theorem 2 above answers a question posed there.) It also leads, less immediately, to similar classification for Z_{p^r} actions on spheres.

Finally, it is possible to analyze the obstruction to concording a non-locally linear action with given fixed set to a locally linear one. The obstructions are mod 2-cohomology classes, introduced in §3, which we call Rothenberg classes after the coefficients of the cohomology in which they lie. They are used to complete the discussion of $BSRN_k^G$.

In future papers we shall use the results developed here to analyze group actions on general manifolds, far removed from disks, spheres, and homology collars, and shall also deal with continuous groups.

Some of the results of this paper were described in the survey [38]. (This paper is a simplification of reference [CW2] of that paper.)

The second author would like to thank the mathematics department of Notre Dame for their hospitality during a useful visit many years ago and the Courant Institute for its continuing support.

1. Actions on bundles

This section analyzes partially the (unstable) difference between equivariant and unequivariant bundle theories to complete the proof of Theorem 1 in the locally linear case. These results are also important for other problems in the theory of PL group actions (see, e.g., [12], [38], [41]). Here we work in the PL-locally smoothable (or locally linear) category, which means that in some equivariant triangulation all orbits are PL equivariantly homeomorphic to linear orbits. It is well known that all smooth actions have such a structure, but this is not the most general PL action. For instance, in this category all fixed point sets are necessarily manifolds, while coning often produces PL actions with nonmanifold fixed point sets. However, even if all fixed point sets are submanifolds, the action need not be locally linear. (We shall analyze this latter type of PL action in §3.) Still the locally linear category is flexible enough to permit geometric constructions yet rigid enough to permit the usual tools of, say, regular neighborhood theory. (This is different from the topological locally linear category which has neither existence nor uniqueness of regular neighborhoods.) In fact this local rigidity is necessary for Reidemeister torsion calculations which seem central to even the existence proof for odd order groups and is certainly necessary for the classification results.

If M is a manifold, we use the simplicial groups (rather Δ-groups [32] $\widetilde{SPL}(M)$ and $\widetilde{\mathrm{Aut}}(M)$, whose k-simplices consist of orientation preserving PL homeomorphisms (autohomotopy equivalences) $\Delta^k \times M \to \Delta^k \times M$ preserving blocks and homotopic to the identity. In our examples, $M = S^n/\rho$, where ρ is a free representation of a finite group G (the putative normal representation to the fixed point set). The relevance here is that:

Proposition (*see, e.g.*, [32]). *$B\widetilde{SPL}(S^n/\rho)$ is the classifying space of n-dimensional oriented equivariant "abstract regular neighborhoods" which are locally modeled on the representation ρ.*

Of course there is a map $B\widetilde{SPL}(S^n/\rho) \to B\widetilde{SPL}(S^n)$ obtained by forgetting the action on the regular neighborhood, or equivalently lifting homeomorphisms via covering space theory. It would be useful to have a splitting of this map, but that does not exist in general; this can be seen from calculations of homotopy groups. However, we will show:

Theorem. *If $\rho: G \to SO(n+1)$ is a free representation, then $B\widetilde{SPL}(S^n/\rho)[1/|G|] \to B\widetilde{SPL}(S^n)[1/|G|]$ splits.*

Moreover, we will identify the fiber away from $|G|$ so as to classify actions on a regular neighborhood. The methods used include blocked surgery theory [24], [5], Sullivan's analysis of F/PL (see, e.g., [20]), and calculations of surgery groups [36].

Our starting point is the identification of the fiber

$$S\widetilde{\mathrm{Aut}}(S^n/\rho)/\widetilde{SPL}(S^n/\rho)$$

of $B\widetilde{SPL}(S^n/\rho) \to BS\widetilde{\mathrm{Aut}}(S^n/\rho)$ with the identity component of the (simple) structure space of S^n/ρ, $\mathbf{S}_0(S^n/\rho)$. (The other components correspond to actions locally modeled on PL representations simple homotopy equivalent to ρ; see [5]). We will henceforth ignore problems involving components. Now we have the surgery fibration structure sequence [36]

$$\mathbf{S}(S^n/\rho) \to (F/PL)^{(S^n/\rho)} \to \mathbf{L}^s_n(G),$$

where $\mathbf{L}^s_n(G)$ is the nth Quinn (simple) surgery space. (Denote the reduced version of this space by $\widetilde{\mathbf{L}}^s_n(G)$.) Using the fact that $\mathbf{S}(D^n)$ is contractible (Poincaré Conjecture) it follows that $\mathbf{S}(S^n/\rho) \sim \mathbf{S}(\mathring{S}^n/\rho)$, where $\mathring{S}^n/\rho$ denotes S^n/ρ punctured by removing an open disk, yielding a fibration

$$\widetilde{\mathbf{L}}^s_{n+1}(G) \to \mathbf{S}(S^n/\rho) \to (F/PL)^{S^n/\rho}.$$

This is already interesting for $n \geq 2$, $G =$ trivial group, where one obtains $S\widetilde{\mathrm{Aut}}(S^n)/\widetilde{SPL}(S^n) \simeq F/PL$ which is the Haefliger-Casson-Sullivan stability theorem under slight disguise. Elsewhere we will discuss equivariant generalizations of this theorem (and its failure to extend in general) and various applications of these.

Now there is a commutative diagram induced by passing to covering spaces:

$$\begin{array}{ccccc} \widetilde{\mathbf{L}}^s_{n+1}(G) & \longrightarrow & \mathbf{S}(S^n/\rho) & \longrightarrow & (F/PL)^{\mathring{S}^n/\rho} \\ \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & \mathbf{S}(S^n) & \longrightarrow & F/PL \end{array}$$

Claim. $(F/PL)^{\mathring{S}^n/\rho} \to F/PL$ is a homotopy equivalence away from $|G|$ so that the fiber of $\mathbf{S}(S^n/\rho) \to \mathbf{S}(S^n)$ is $\widetilde{\mathbf{L}}^s_{n+1}(G)$ away from $|G|$.

The proof is a straightforward calculation. Evaluation at a base point $*$ produces a fibration

$$(F/PL\,,\,*)^{(\mathring{S}^n/\rho\,,\,*)} \to (F/PL)^{(\mathring{S}^n/\rho)} \to F/PL\,,$$

the fiber being weakly contractible $1/|G|$ by the Federer spectral sequence [15] $E_2^{p,q} = H^p(Y\,;\,\pi_q(X)) \Rightarrow \pi_{p-q}((X\,,\,*)^{(Y\,,*)})$ for the homotopy groups of (components of) function spaces of based maps. We now obtain:

Proposition. *Away from $|G|$ there is a fibration*

$$\widetilde{\mathbf{L}}^s_{n+1}(G) \to B\widetilde{SPL}(S^n/\rho) \to B\widetilde{SPL}(S^n)\,.$$

Proof. Another two applications of the Federer spectral sequence shows that

$$BS\widetilde{\mathrm{Aut}}(S^n/\rho)[1/|G|] \to BS\widetilde{\mathrm{Aut}}(S^n)[1/|G|]$$

is a homotopy equivalence, and the result follows from the previous claim. q.e.d.

To finish the proof of Theorem 1 for $\widetilde{BSPL}^G_*$ we show that this fibration is a product away from $|G|$ by giving a section to $\widetilde{\mathbf{L}}_{n+1}(G)$. The homotopy structure of $\widetilde{\mathbf{L}}_{n+1}(G)$ has been established by L. Jones and (independently) by L. Taylor and B. Williams. Away from 2 it is a wedge of BO spectra and at 2 it is a wedge of Eilenberg-Mac Lane (EM) spectra. We will not use their results, but rather rederive them partially in producing a map from $B\widetilde{SPL}(S^n/\rho)$ to the BO and EM spectra models in such a way that composition with the map $\widetilde{\mathbf{L}}^s_{n+1}(G) \to B\widetilde{SPL}(S^n/\rho)$ will be a homotopy equivalence $[1/|G|]$. The map to the EM spectra at 2 for G odd involves a more precise description below of the image of the multisignatures that arise from $B\widetilde{SPL}(S^n/\rho)$.

Construction of the map. **A**: $B\widetilde{SPL}(S^n/\rho) \to \bigvee \sum^i BO(\widetilde{L}_i(G))[\frac{1}{2}|G|]$. By $BO(\Lambda)$ we mean BO with coefficients in Λ (e.g., smash with a Moore spectrum). The coefficient groups $\widetilde{L}_i(G)[\frac{1}{2}]$ are quite easy to understand.

For i odd they vanish and for i even they are closely related to representation theory via the multisignature [37]. For definiteness we then just construct "half" of the map

$$\mathbf{A}\colon B\widetilde{SPL}(S^n/\rho) \to BO(\widetilde{RO}(G))\left[\frac{1}{2}|G|\right],$$

where $\widetilde{RO}(G)$ is the ordinary real representation ring modulo the regular representation. According to Sullivan's "Connor-Floyd" theory [34] this is the same as constructing a homomorphism

$$\Omega^{SO}(B\widetilde{SPL}(S^n/\rho)) \otimes_{\Omega^{SO}(*)} Z[1/2] \to \widetilde{RO}(G)\left[\frac{1}{2}|G|\right],$$

where $Z[\frac{1}{2}]$ is viewed as a $\Omega^{SO}(*)$ module by multiplication by signature. Such a homomorphism is provided by the Atiyah-Singer invariant of free actions. If

$$[M, f] \in \Omega_k^{SO}(B\widetilde{SPL}(S^n/\rho),$$

then there are associated PL block bundles:

$$\begin{array}{ccccc} S^n & \longrightarrow & \widetilde{E} & & \\ & \searrow & & \searrow & \\ & & (S^n/\rho) & \longrightarrow & E \\ & & & & \downarrow \\ & & & & M \end{array}$$

Note that the cover $\widetilde{E}$ bounds (cone down to M). Therefore $|G|^l E$ bounds in $\Omega_{n+k}^{SO}(BG)$ for some l. The multisignature of a coboundary divided by $|G|^l$ gives an element of $\widetilde{RO}(G)[1/|G|]$. Subtract now the signature of the disk bundle (obtained by coning the sphere bundle forgetting the group action) times the trivial representation. A standard argument shows that l and the coboundary do not change the element obtained. It is also easy to see (using homological triviality of bundle automorphisms) that the construction only depends on the bordism class of $[M, f]$.

Claim. The composite

$$\widetilde{\mathbf{L}}_{n+1}^s(G) \to B\widetilde{SPL}(S^n/\rho) \to \bigvee \sum^i BO(\widetilde{L}_i^s(G))\left[\frac{1}{2}|G|\right]$$

is localization.

The proof of this follows by replacing $B\widetilde{SPL}(S^n/\rho)$ by the structure space $\mathrm{S}(S^n/\rho)$, the opposite of what was done above, and then computing the map on homotopy groups. This last computation just recalls how the Atiyah-Singer invariants change by an action of elements of the Wall group, e.g., by the multisignature which is an isomorphism $\otimes Z[\frac{1}{2}]$. It is the necessary tensoring with $Z[\frac{1}{2}]$ (typically only multiples of four are in the image of the multisignature) that makes the problem at the prime 2 more difficult. We will rely on the deep calculations of [37], taking advantage of the fact that local linearity forced L^s-theory on us. In any case, for G of even order, Theorem 1 is proved (for the locally linear case).

For G of odd order, it is best to concentrate on $G = Z_r$, since a standard argument involving Dress induction [14] shows that this is the critical case. Intuitively, the Atiyah-Singer invariant is replaced by the (main part of the) obstruction of making an odd multiple of E bound a homologically trivial Z_r-manifold.

Construction of the map. B: $B\widetilde{SPL}(S^n/\rho) \to \bigvee EM(L^s_{k+i+1}(G), i)_{(2)}$.

Consider $[M, f] - [M, *]$ which produces a similar situation to the above. Now multiplying by r^l one obtains something that bounds, and consider the symmetric signature [25] of the coboundary as an algebraic Poincaré chain complex with boundary over the argumentation ideal $\mathbb{I}$ of $R[Z_r]$. By homological triviality again, the boundary is contractible; so a priori, one obtains an element in $L_h^{n+k+1}(\mathbb{I})$. Note that as $\frac{1}{2} \in R$, the symmetric signature group of Ranicki can be identified with Wall's surgery group, i.e., $L^i_x = L^x_i$, $x = s$ or h. Moreover, it is easy to see that the simplicity obstruction of this chain complex [11, §2], i.e., the image under the Rothenberg sequence to the cohomology of the Whitehead group, is given by the Reidemeister torsion of the boundary. Since Z_r action preserves blocks and always has the same simple homotopy type for the different blocks, there is a product formula for this [11, §5]; hence by taking the reduced version $[M, f] - [M, *]$, the Reidemeister torsion vanishes. Thus this element is in the group $L_s(\mathbb{I})$. Now $\mathbb{I}$ is a product of type O factors, so this group is detected by multisignature and thus there is a unique preimage. Dividing by r gives an element of $L_s^{n+k+1}(\mathbb{I}) \otimes Z_{(2)}$. Again, it is well defined and yields a homomorphism

$$\Omega_k(B\widetilde{SPL}(S^n/\rho)) \otimes_{\Omega_*(*)} \to L_s^{n+k+1}(\mathbb{I}) \otimes \mathbb{Z}_2$$

which yields

$$\mathbf{B}: B\widetilde{SPL}(S^n/\rho) \to \bigvee EM(\widetilde{L}^s_{n+i+1}(\mathbb{I}), i)_{(2)}.$$

However, Wall shows [36] that $\widetilde{L}^s_i(Z_k)$ and $L^s_i(\mathbb{I})$ are isomorphic with identical ranges of multisignatures so that the argument can be completed as before, finishing the proof of the theorem.

Remark. To get actual choice of classes, rather than just show they exist, one must repeat the same argument with $\mathbb{Z}_n$-manifolds as in, for example, the discussion of F/PL in [23].

2. Hard extensions across homology collars and applications

In this section we prove Theorem 2 of the introduction in the locally linear case. There are two ingredients: the bundle theory of §1 and the special case where the submanifold $L = \varnothing$. This special case was first proven by A. Assadi and W. Browder (jointly and unpublished) and the second author [39]. We restate it here for the readers' convenience.

Theorem. *Suppose* $(W^n, \partial_+, \partial_-)$ *is a* $Z_{(|G|)}$ *homology collar (i.e.,* $H_*(W, \partial_\pm; Z_{(|G|)}) = 0$*), and* G *acts freely and* $Z[1/|G|]$ *homologically trivially on* ∂_-. *Suppose further* $\pi_1(\partial_+) = \pi_1(W) = 0$ *and* $n \geq 6$. *Then the* G *action on* ∂_- *extends to such a homologically trivial* G *action on* W *if and only if* $\sum(-1)^i\sigma(|H_i(W, \partial_-)|) = 0 \in \widetilde{K}_0(ZG)$.

Extensions differ by an element of $\mathrm{Wh}(G)$.

This statement differs from that in [39] only in that there ∂_- was supposed simply connected. However, an examination of the proof given there shows that this hypothesis was never used (except to reassure the author!). In any case one can use the fact $H_1(\partial_-; Z_{(|G|)}) = 0$ to do a "+ construction" equivariantly with respect to G to get the data needed for the Zabrodsky mixing of that paper.

Theorem 2 now follows from Theorem 1. After extending the Atiyah-Singer classes one can extend $1/|G|$ the map $L \to B\widetilde{SPL}_k^G$ to a map $K \to B\widetilde{SPL}_k^G$. At $|G|$, $L = K$ so there is no difficulty in the extension. We now have an action on $\partial_- \times I \cup \mathrm{Nbd}(L)$. The complement of this union is a homology collar and the action already given on its "bottom boundary" is free. Homological triviality follows from a Mayer-Vietoris argument, and simple connectivity follows from the codimensionality hypothesis. q.e.d.

We record two corollaries.

Corollary 1. *Let* $\rho: G \to SO(k)$, $k > 2$, *be a semifree representation. A submanifold* F^n *of* D^{n+k} *is the fixed set of a semifree locally linear* PL G *action with normal representation* ρ *if and only if* $\widetilde{H}_*(F; Z_{(|G|)}) = 0$ *and* $\sum(-1)^i\sigma(|H_i(F; Z)|) = 0 \in \widetilde{K}_0(ZG)$.

Actions are classified up to an element of $\mathrm{Wh}(G)$ by the Atiyah-Singer class in $[F: \widetilde{\mathbf{L}}^s(G)1/|G|]$ (and every element is the Atiyah-Singer class of some action).

The proof is trivial from Theorem 2.

Corollary 2. *If* p *is a prime, and* Σ^n *is a* Z_p *homology subsphere of* S^{n+2k}, $k > 1$, *then there is a locally linear* Z_{p^r} *action on* S^{n+2k} *with fixed set* Σ.

Although the proof of this is more complicated, we shall be brief since the second author has classified the semifree actions for arbitrary groups as an application of the ideas of this paper and Jim Davis' method of detecting surgery obstructions [14]. One punctures Σ, makes it the fixed set on the disk, and then examines the action on the boundary. By [29] this action on a sphere with fixed set a subsphere is determined (PL) by a Whitehead torsion. However, our action is only well defined up to concordance which changes the Rothenberg-Sondow elements by a norm. This leads to a problem involving $H^*(Z_2; \mathrm{Wh}(Z_{p^r}))$. For a p odd [11] immediately leads to the vanishing. For $p = 2$ one tries to solve the problem for $Z_{p^{r+1}}$ and retreats to Z_{p^r} and makes use of the vanishing of the transfer on Rothenberg groups [7].

3. Rothenberg classes

This section studies actions that are not necessarily locally linear but do have manifold fixed sets. An example of this situation occurred in the previous section where it was fairly easy to get something to be a fixed set in a sphere but local linearity required the vanishing of an additional obstruction. In general we shall see that there are ordinary cohomology class obstructions to concording an action with fixed set F to a locally linear action. The classes lie in $H^i(F; H^{n-i}(Z_2; \mathrm{Wh}(G)))$ and we call them Rothenberg classes. (One can define these much more generally, for nonsemifree actions, but then they lie in a sheaf cohomology group with local stalk the Tate cohomology of the Rothenberg-Illman Whitehead group of the isotropy.)

The Atiyah-Singer classes appropriate to locally linear actions lie in $[F: \widetilde{\mathbf{L}}^s(G)[1/|G|]]$. The ones related to PL actions with manifold fixed set lie in $[F: \widetilde{\mathbf{L}}^h(G)[1/|G|]]$. There is an obvious connection between the difference of receiving groups for these classes and the Rothenberg classes via the Rothenberg sequence [33]. The fact that the classes lie in cohomology is due to the fact that, according to that sequence, the fiber of the map $\mathbf{L}^s(G) \to \mathbf{L}^h(G)$ is 2-local and then, by general principles, Eilenberg-Mac Lane. (It is a module over $L^*(\mathbb{Z})$ which is Eilenberg-Mac Lane at 2.)

Proposition. *There is a classifying space for concordance classes of equivariant k-codimension regular neighborhoods of manifolds.*

The proof of this is the same as the corresponding result for nonlocally flat codimension two embeddings [8], and will hence be deleted (see also [22]). A little thought shows that the components are in a 1-1 correspondence with concordance classes of the free PL G actions on S^{n-1}. Thus $BRN_k(G) \to \bigcup_\rho BRN_k(\rho)$ is a homotopy equivalence.

An analysis as in §1 for $BSRN_k(G)$ is feasible. This works very well for the fiber of $BSRN_k(G) \to BS\,\mathrm{Aut}(S^n/\rho)$. The trouble with defining Atiyah-Singer classes directly is due to the difficulty in understanding $L_n^h(G)_{(2)}$. We shall compare $BSRN_k(G)$ to $B\widetilde{SPL}_k(G)$ and leave the deduction of the existence of relevant L^h-Atiyah-Singer classes to the reader. (An alternative is to refer to [40] which gives a relevant integrability theorem for ρ-invariants in the nonlocally linear case.)

One has fibrations:

$$\begin{array}{ccccc} \mathbf{S}^s(S^{k-1}/\rho) & \longrightarrow & B\widetilde{SPL}_k(\rho) & \longrightarrow & BS\,\mathrm{Aut}(S^{n-1})/\rho) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{S}^h(S^{k-1}/\rho) & \longrightarrow & BSRN_k(\rho) & \longrightarrow & BS\,\mathrm{Aut}(S^{k-1}/\rho) \end{array}$$

Consequently, the fiber of $B\widetilde{SPL}_k(\rho) \to BSRN_k(\rho)$ is the same as that of the simple and homotopy structure spaces. These latter can be analyzed as before:

$$\begin{array}{ccccc} \mathbf{L}_k^s(G) & \longrightarrow & \mathbf{S}^s(S^{k-1}/\rho) & \longrightarrow & (F/PL)^{S^{k-1}/\rho} \\ \downarrow & & \downarrow & & \downarrow \approx \\ \mathbf{L}_k^h(G) & \longrightarrow & \mathbf{S}^h(S^{k-1}/\rho) & \longrightarrow & (F/PL)^{S^{k-1}/\rho} \end{array}$$

We shall denote the fiber of $\mathbf{L}_k^s(G) \to \mathbf{L}_h^s(G)$ by $A_k(G)$. There is a well-known connection between $\pi_n(A_k(G))$ and the cohomology of the Whitehead group.

Theorem. *There is a fibration*

$$B\widetilde{SPL}_k(G) \to B\widetilde{SRN}_k(G) \to BA_k(G) = \prod_{i=0}^{\infty} K(H^{i+k}(Z_2; \mathrm{Wh}(G), i)).$$

Proof. We shall construct cohomology classes, which it is then easy to see produce a classifying map for the fibration. For each component of

$BS\widetilde{RN}_k(G)$ pick a specific free G action on S^{k-1}. There is a homotopy equivalence

$$BS\widetilde{\text{Homeo}}(S^{k-1}/G \times Q) \to BS\widetilde{\text{Aut}}(S^{k-1}/G),$$

where Q is the Hilbert cube. (See [13] for background.) By choosing a homotopy inverse one gets a well-defined *simple* homotopy type for any map $M \to BS\widetilde{\text{Aut}}(S^{k-1}/G)$. We get an l-dimensional cohomology class by taking the torsion of the homotopy equivalence from the pullback of the boundary sphere bundle over F to a submanifold of dimension l to the associated Hilbert manifold bundle over that submanifold. By the time one considers the class in Tate cohomology, it is independent of all choices. Surgery theory, and tracing through the above fibrations, complete the proof of the theorem. q.e.d.

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